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Learning to Learn

The Art of Doing Science and Engineering

Session 15: Digital Filters II

Digital Filters

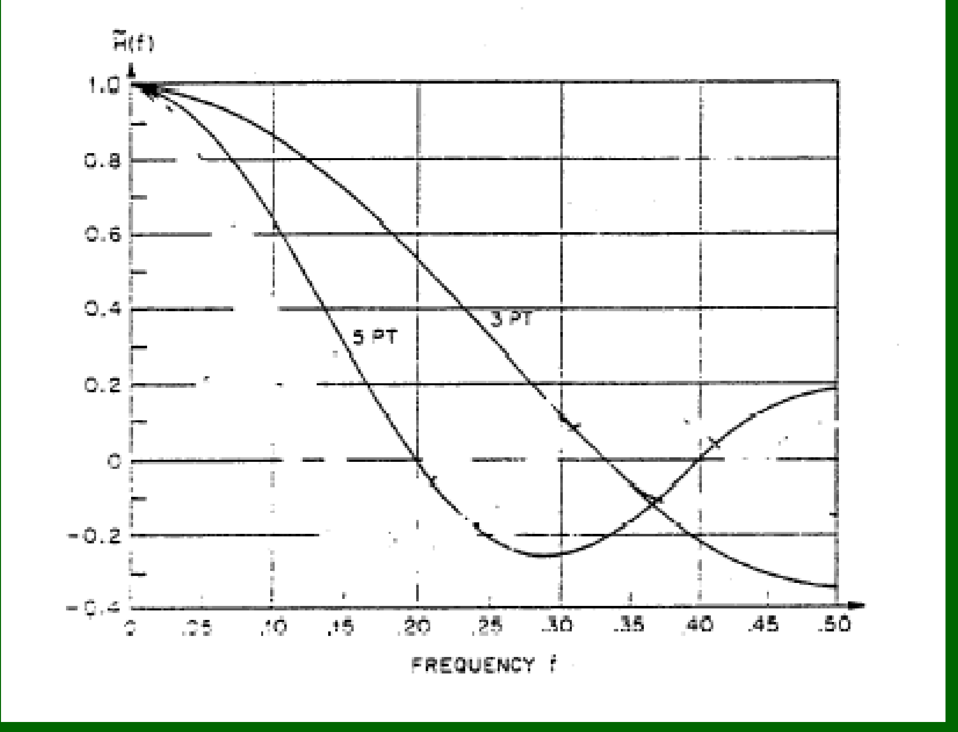
Endless mistake to think that something new is just like the past.

Earliest filters smoothed first by 3’s and then by 5’s.

Removes frequencies from a stream of numbers.

Digital Filters

$$H(\omega) = \{\sin(3/2)\omega / 3\sin(\omega/2)\}$$



Design of a Simple Filter

In theory angular frequency is used, but in practice rotations are used.

$$f = \omega / 2\lambda \quad (-1/2 < f < 1/2)$$

$$y_n = a u_{n-1} + b u_n + a u_{n+1}$$

Substituting in the eigenfunction

$$\exp\{2\lambda i f n\}$$

$$H(f) = b + 2a \cos 2\lambda f$$

$$y_n = (1/2)[u_{n-1} + u_n + u_{n+1}]$$

Design of Simple Filters

Output of the filter is the sun of three consecutive inputs divided by 2, and the output is opposite the middle input value.

n	1/6	1/3	sum
0	1	1	2
1	1/2	−1/2	0
2	−1/2	−1/2	−1
3	−1	1	0
4	−1/2	−1/2	−1
5	1/2	−1/2	0
6	1	1	2
7	1/2	−1/2	0
8	−1/2	−1/2	−1
...	...	...	...

Design of Simple Filters

The filter decomposes the input signal into all its frequencies, multiplies each frequency by its corresponding eigenvalue, the transfer function, and then adds all the terms together to give the output.

Ideally we want a transfer function that has a sharp cutoff between the frequencies it passes and those it stops.

Gibbs’ Phenomena

Theorem: If a series of continuous function converges uniformly in a closed interval then the limit function is continuous.

- Story- Michelson noticed an overshoot when going from coefficients of a Fourier Series back to a function. When he asked local mathematicians about the problem, they said it was his computer.
- Gibbs, from Yale, listened and looked into the matter.

Gibbs’ Phenomena

Simplest direct approach is to expand a standard discontinuity into a Fourier Series of a finite number of terms.

After rearranging, then find the location of the first maximum and finally the corresponding height of the functions there.

Many people had the opportunity to discover the Gibb’s phenomena, and it was Gibbs that made the effort.

Hamming Assertion

There are opportunities all around and few people reach for them.

Pasteur said, “Luck favors the prepared mind.”

This time the person who was prepared to listen and help a first class scientist in his troubles (Gibbs) got the fame.

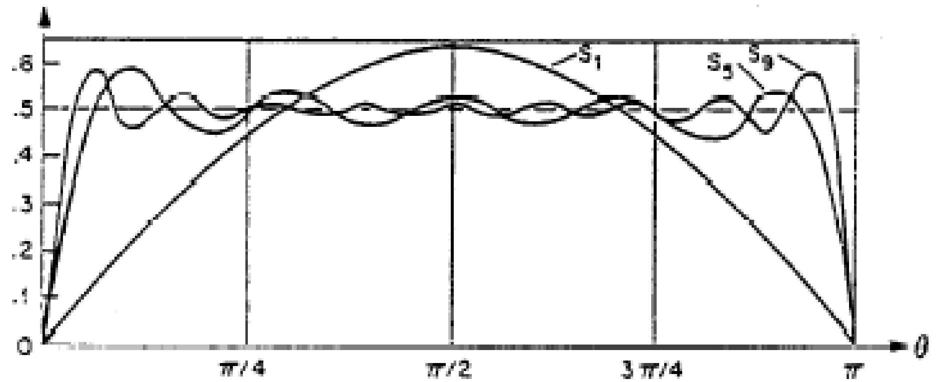
Cauchy’s Textbooks

Contradictions

- A convergent series of continuous functions converge to a continuous function.
- The Fourier expansion of a discontinuous function.

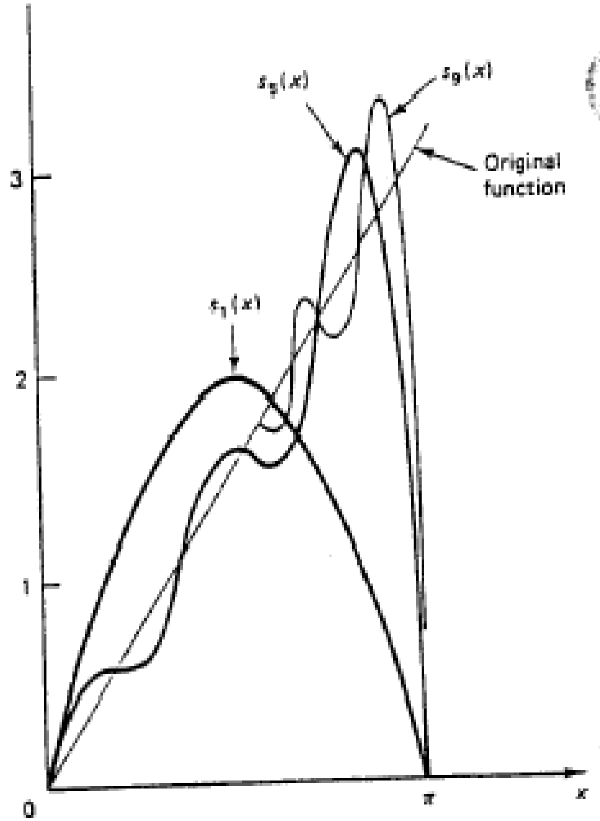
The concept of uniform convergence. The overshoot of the Gibb’s phenomena occurs for any series of continuous functions, not just Fourier Series.

Gibb’s Phenomena



PARTIAL SUMS S_1, S_5, S_9 FOR RECTANGULAR PULSE

Figure 15-2

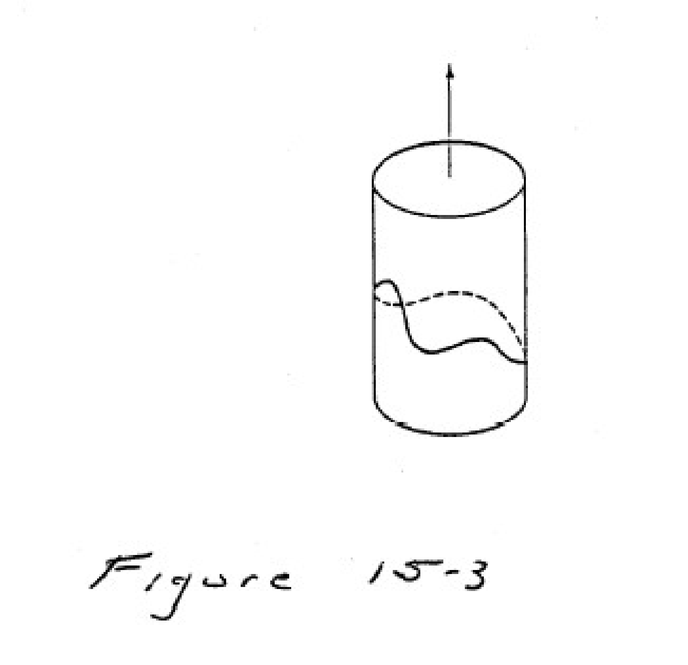


Another Feature of a Fourier Series

If the function exists then the coefficients fall off like 1/n.

If the function is continuous and the derivative exists then the coefficients fall off like $1/n^2$.

If the first derivative is continuous and the second derivative exists then they fall off like $1/n^3$, etc.



Lanczos' Window

Set up the integral for the averaging

$$(N/2\lambda)\int_{x+\lambda/N}^{x-\lambda/N} g(s)\,ds$$

$$=(N/2\lambda)\int[\{a_0/2+\sum_1^N\{a_k\cos ks+b_k\sin ks]\}ds]$$

$$=a_0/2+(N/2\lambda)\sum_1^N[a_k/k\sin ks-(b_k/k)\cos ks]\Big|_{x-\lambda/N}^{x+\lambda/N}$$

$$=a0/2+\sum_1^N[a_k\cos kx+b_k\sin kx][\{\sin(\lambda k/N)/(\lambda k/N)\}]$$

$$\sigma(N,k)=\sin(\lambda k/N)/(\lambda k/N)$$

Effects of Lanczos' Window

Reduce Overshoot

- Reduced to 0.01189, a factor of 7
- Reduce first minimum to 0.00473, a factor of 10
- Significant but not a complete reduction of the Gibbs' phenomenon.
- At discontinuity the truncated Fourier expansion takes on the mid-value of the two limits, one from each side.

Transfer Function

Another Approach-Modified Series

$$g(x) = \sum_{-\infty}^{\infty} [c_k \exp\{ikx\}]$$

$$h(x) = \sum_{-\infty}^{\infty} [d_m \exp\{imx\}]$$

$$g(x)h(x) = \sum_{n=-\infty}^{\infty} \{\sum_{k=-\infty}^{\infty} [c_k d_{n-k}]\}\exp\{inx\}$$

$$\sum_{-K}^{K} [c_k d_{n-k}]$$

Digital Filter

A filter is the convolution of one array by another, and that in turn is the multiplication of the corresponding functions.

$$\exp\{iNx\}+\exp\{i(N-1)x\}+\ldots+\exp(0)+\ldots+\exp\{iNx\}$$

$$\exp\{iNx\}[1-\exp\{i(2N+1)x\}]/[1-\exp\{ix\}](2N+1)$$

$$=\sin\{(N=1/2x)/(2N+1)\sin(x/2)\}$$

Digital Filter

Simple modification of Lanczos' Window by changing the outer two coefficients from 1 to ½ produce a better window.

$$(\sin x)/x$$

$$w_k = (1 + \cos \pi k / N)/2$$

Writing out in exponentials is

$$1/4, 1/2, 1/4$$

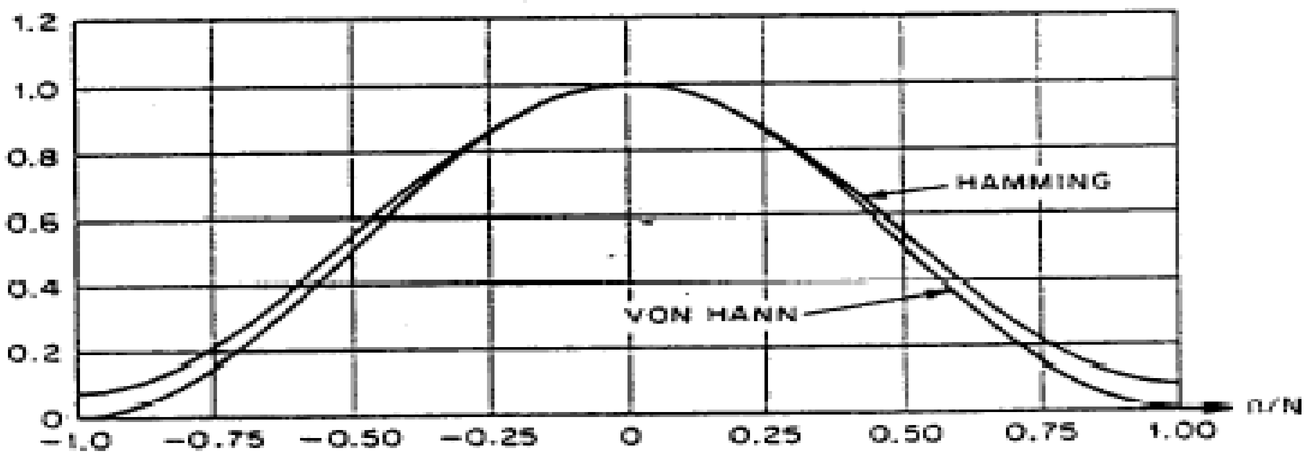
Writing out in von Hann "raised cosine"

$$1/4, 1/2, 1/4$$

Hamming window is "raised cosine on a platform"

$$0.23, 0.54, 0.23$$

Digital Filter



WEIGHT FACTORS FOR HAMMING AND VON HANN WINDOWS

Figure 15-4